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RESEARCH ARTICLES

Weighted Central BMO Type Space Estimates for Commutators of p-Adic Hardy-Cesàro Operators

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Abstract—The aim of this paper is to give some sufficient conditions for the boundedness of commutators of p-adic Hardy-Cesàro operators with symbols in weighted central BMO type spaces on the Herz spaces, Morrey spaces and Morrey-Herz spaces with both the Muckenhoupt and power weights.

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1. INTRODUCTION

It is well known that the theory of functions from $\mathbb{Q}_p$ into $\mathbb{C}$ plays an important role in p-adic quantum mechanics, the theory of p-adic probability in which real-valued random variables have to be considered to solve covariance problems (see, for example, [12, 17, 28] and references therein). In recent years, p-adic analysis has got a lot of attention by its important application in mathematical physics. In particular, there is an increasing interest in the study of p-adic wavelet analysis and p-adic harmonic analysis, for instance, p-adic Hardy, p-adic Hardy-Cesàro, p-adic Hausdorff operator as well as their applications (see [1, 4–7, 10, 14, 18, 20, 25, 29–31] and references therein).

In 1984, Carton-Lebrun and Fosset [3] studied the weighted Hardy-Littlewood average operator as follows

$$\mathcal{H}_\varphi(f)(x) = \int_0^1 \varphi(y)f(yx)dy, \;\; x \in \mathbb{R}^n,$$

where $\varphi : [0,1] \to [0,\infty)$ is a measure function. In 2001, J. Xiao [32] established the necessary and sufficient conditions for the boundedness of $\mathcal{H}_\varphi$ and obtained its norm on the Lebesgue and BMO spaces. Next, in 2014, Chuong and Hung [9] introduced the Hardy-Cesàro operator defined by

$$\mathcal{C}_{\varphi,s}(f)(x) = \int_0^1 \varphi(y)f(s(y)x)dy, \;\; x \in \mathbb{R}^n,$$

where $\varphi : [0,1] \to [0,\infty)$ and $s : [0,1] \to \mathbb{R}$ are measurable functions. On the p-adic fields, the Hardy-Cesàro operator introduced by Hung [14] as follows

$$\mathcal{C}^p_{\varphi,s}(f)(x) = \int_{\mathbb{Z}_p^*} \varphi(y)f(s(y)x)dy, \quad x \in \mathbb{Q}_p^n,$$

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where φ is a locally integrable function on $\mathbb{Z}_p^*$ and $s : Z_p^* \to \mathbb{Q}_p$ is a measurable function. Obviously, by setting $s(y) = y$, the operator $\mathcal{C}^p_{\varphi,s}$ then reduces to the p-adic weighted Hardy-Littlewood average operator studied by Rim and Lee [24] as follows

$$\mathcal{H}^p_\varphi f(x) = \int_{\mathbb{Z}_p^*} f(yx)\varphi(y)dy, \quad x \in \mathbb{Q}_p^n.$$

Especially, for $n = 1$ and $\varphi = 1$, the operator $\mathcal{H}^p_\varphi$ reduces to p-adic Hardy operator defined by

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p} \int_{|y|_p \le |x|_p} f(y)dy.$$

For further information on the p-adic Hardy-Cesàro operators as well as their applications, one can be found in [4, 7, 14, 29, 31] and therein references. Remark that the operator $\mathcal{H}^p_\varphi$ is closely connected with solution of some pseudo-differential equations on p-adic fields posed by Kochubei [19] as follows

$$\begin{cases} D^\alpha v + a(|x|_p)v = f(|x|_p), & x \in \mathbb{Q}_p, \\ v(0) = 0, \end{cases}$$

where D^α is the Vladimirov operator of order α. The solution of this problem is found in terms of the form $v = \mathcal{R}^p_\alpha(u)$, where $\mathcal{R}^p_\alpha$ is the p-adic Riemann-Liouville fractional operator defined by

$$\mathcal{R}^p_\alpha(u)(x) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int\limits_{|y|_p \le |x|_p} \left(|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) u(y)dy. \tag{1.1}$$

It is easy to see that

$$\mathcal{R}^p_\alpha(u)(x) = \left(\mathcal{H}^p_{\varphi_1} u(x) - \mathcal{H}^p_{\varphi_2} u(x)\right) |x|_p^\alpha,$$

where

$$\varphi_1(y) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} |1 - y|_p^{\alpha-1}, \quad \text{and} \quad \varphi_2(y) = \varphi_1(1 - y).$$

In recent years, the weighted Hardy-Littlewood average operators, Hardy-Cesàro operators and Hausdorff operators and their commutators have been significantly developed into different contexts (see [2, 6, 7, 9–11, 22, 23, 25]). As is well known, the theory of commutators plays an important role in the study of the regularity of solutions to partial differential equations. In this paper, we discuss the commutators of Coifman-Rochberg-Weiss type of p-adic Hardy-Cesàro operators as follows

$$\mathcal{C}^{p,b}_{\varphi,s}(f)(x) = \int_{\mathbb{Z}_p^*} \varphi(y)\Big(b(x) - b(s(y)x)\Big) f(s(y)x)dy, \quad x \in \mathbb{Q}_p^n.$$

In case $s(y) = y$, $\mathcal{C}^{p,b}_{\varphi,s}$ will reduce to the commutator of Hardy-Littlewood operators $\mathcal{H}^{p,b}_\varphi$ as follows

$$\mathcal{H}^{p,b}_\varphi(f)(x) = \int_{\mathbb{Z}_p^*} \varphi(y)\Big(b(x) - b(yx)\Big) f(yx)dy, \quad x \in \mathbb{Q}_p^n.$$

The main purpose of this paper is to establish some sufficient conditions for the boundedness of the commutator $\mathcal{C}^{p,b}_{\varphi,s}$ with symbols in weighted central BMO type spaces on the p-adic Herz spaces, p-adic Morrey spaces and p-adic Morrey-Herz spaces associated with both power weights and the Muckenhoupt weights. As a consequence, we also have the boundedness of commutators $\mathcal{H}^{p,b}_\varphi$ on such spaces.

Our paper is organized as follows. In Section 2, we present some notations and definitions of p-adic analysis, the class of Muckenhoupt weights on the p-adic field as well as some p-adic weighted function spaces such as p-adic Morrey, Herz, Morrey-Herz and central BMO spaces. Our main results are given and proved in Section 3.

2. SOME NOTATIONS AND DEFINITIONS

Let us give a brief introduction on p-adic analysis. For a more complete information to p-adic analysis, see [17, 28] and the references therein. For a prime number p, denote by $\mathbb{Q}_p$ the field of p-adic numbers. This field is the completion of the field of rational numbers with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if $x \neq 0$ is an arbitrary rational number with the unique representation $x = p^k \frac{m}{n}$, where m, n are not divisible by p, $k \in \mathbb{Z}$, then $|x|_p = p^{-k}$. It is easy to verify that this norm has the following properties:

(i) $|x|_p \geq 0, \ \ \forall x \in \mathbb{Q}_p, |x|_p = 0 \Leftrightarrow x = 0$;

(ii) $|xy|_p = |x|_p |y|_p, \ \ \forall x, y \in \mathbb{Q}_p$;

(iii) $|x + y|_p \leq \max(|x|_p, |y|_p), \ \forall x, y \in \mathbb{Q}_p$, and when $|x|_p \neq |y|_p$, we have $|x + y|_p = \max(|x|_p, |y|_p)$. Moreover, any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^k (x_0 + x_1 p + x_2 p^2 + \cdots), \tag{2.1}$$

where $k \in \mathbb{Z}, \ x_m = 0, 1, ..., p-1, \ x_0 \neq 0, \ m = 0, 1, ...$ This series, of course, converges in the p-adic norm since $|x_m p^k|_p \leq p^{-k}$.

Let $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$. The p-adic norm of $\mathbb{Q}_p^n$ is defined as follows

$$|x|_p = \max_{1 \leq i \leq n} |x_i|_p, \quad x = (x_1, ..., x_n). \tag{2.2}$$

Let

$$B_k(a) = \left\{ x \in \mathbb{Q}_p^n : |x - a|_p \leq p^k \right\}$$

be a ball of radius p^α with center at $a \in \mathbb{Q}_p^n$. Similarly, denote by

$$S_k(a) = \left\{ x \in \mathbb{Q}_p^n : |x - a|_p = p^k \right\}$$

the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^α. Denote $B_k = B_k(0), S_k = S_k(0)$. Thus for any $x_0 \in \mathbb{Q}_p^n$ we get $x_0 + B_k = B_k(x_0)$ and $x_0 + S_k = S_k(x_0)$. Especially, we denote $\mathbb{Z}_p$ instead of B_0, $\mathbb{Z}_p^* = B_0 \setminus \{0\}$ in $\mathbb{Q}_p$, $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ and χ_k be the characteristic function of the sphere S_k.

It is known that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. This measure is unique by normalizing dx such that

$$\int_{B_0} dx = |B_0| = 1,$$

where $|B|$ denotes the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. For $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$, we have

$$\int_{\mathbb{Q}_p^n} f(x) dx = \lim_{k \to +\infty} \int_{B_k} f(x) dx = \lim_{k \to +\infty} \sum_{-\infty < m \leq k} \int_{S_m} f(x) dx.$$

In the case $f \in L^1(\mathbb{Q}_p^n)$, one may write $\int_{\mathbb{Q}_p^n} f(x)dx = \sum_{m=-\infty}^{+\infty} \int_{S_m} f(x)dx$. By simple calculation, it is easy to obtain that $|B_\alpha(a)| = p^{n\alpha}, \ |S_\alpha(a)| = p^{n\alpha}(1 - p^{-n}) \simeq p^{n\alpha}$, for any $a \in \mathbb{Q}_p^n$. Besides that, we also have $\omega(B_\alpha) \simeq p^{\alpha(n+\gamma)}$ with $\omega(x) = |x|_p^\gamma$ $(\gamma > -n)$.

Let ω be a positive measurable function almost everywhere in $\mathbb{Q}_p^n$. The weighted Lebesgue space $L^q_\omega(\mathbb{Q}_p^n)$ $(0 < q < \infty)$ is defined to be the space of all Haar measurable functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^q_\omega(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^q \omega(x) dx \right)^{1/q} < \infty.$$

The space $L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n)$ is defined as the set of all measurable functions f on $\mathbb{Q}_p^n$ satisfying $\int_K |f(x)|^q\omega(x)dx < \infty$ for any compact subset K of $\mathbb{Q}_p^n$. The space $L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n \setminus \{0\})$ is also defined in a similar way as the space $L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n)$.

Throughout the whole paper, we denote by C a positive geometric constant that is independent of the main parameters, but can change from line to line. Denote $\omega(B)^\lambda = \left(\int_B \omega(x)dx\right)^\lambda$, for $\lambda \in \mathbb{R}$. We also write $a \lesssim b$ to mean that there is a positive constant C, independent of the main parameters, such that $a \le Cb$. The symbol $f \simeq g$ means that f is equivalent to g (i.e. $C^{-1}f \le g \le Cf$). For any real number $\ell > 1$, denote by ℓ' conjugate real number of ℓ, i.e. $\frac{1}{\ell} + \frac{1}{\ell'} = 1$.

Let us give the definition of weighted λ-central Morrey p-adic spaces.

Definition 2.1. *Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The weighted λ-central Morrey p-adic spaces $\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)$ consists of all Haar measurable functions $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n)$ satisfying $\|f\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} < \infty$, where*

$$\|f\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \Big(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q\omega(x)dx\Big)^{1/q}. \tag{2.3}$$

Remark that $\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)$ is a Banach space and reduces to $\{0\}$ when $\lambda < -\frac{1}{q}$.

We also present some definitions of the weighted Herz and Morrey-Herz p-adic spaces.

Definition 2.2. *Let $\beta \in \mathbb{R}, 0 < q < \infty$ and $0 < \ell < \infty$. The weighted Herz p-adic space $K^{\beta,\ell}_{q,\omega}(\mathbb{Q}_p^n)$ is defined as the set of all functions $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n \setminus \{0\})$ such that $\|f\|_{K^{\beta,\ell}_{q,\omega}(\mathbb{Q}_p^n)} < \infty$, where*

$$\|f\|_{K^{\beta,\ell}_{q,\omega}(\mathbb{Q}_p^n)} = \Big(\sum_{k=-\infty}^{\infty} p^{k\beta\ell}\|f\chi_k\|^\ell_{L^q_\omega(\mathbb{Q}_p^n)}\Big)^{1/\ell}. \tag{2.4}$$

Definition 2.3. *Let $\beta \in \mathbb{R}, 0 < q < \infty$ and $0 < \ell < \infty$. The weighted Herz p-adic space $\dot{K}^{\beta,\ell,q}_\omega(\mathbb{Q}_p^n)$ is defined as the set of all functions $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n \setminus \{0\})$ such that*

$$\|f\|_{\dot{K}^{\beta,\ell,q}_\omega(\mathbb{Q}_p^n)} = \Big(\sum_{k=-\infty}^{\infty} \omega(B_k)^{\beta\ell/n}\|f\chi_k\|^\ell_{L^q_\omega(\mathbb{Q}_p^n)}\Big)^{1/\ell} < \infty. \tag{2.5}$$

Definition 2.4. *Let $\beta \in \mathbb{R}, 0 < q < \infty$, $0 < \ell < \infty$ and λ be a non-negative real number. The weighted Morrey-Herz p-adic space is defined by*

$$MK^{\beta,\lambda}_{\ell,q,\omega}(\mathbb{Q}_p^n) = \Big\{f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{MK^{\beta,\lambda}_{\ell,q,\omega}(\mathbb{Q}_p^n)} < \infty\Big\},$$

where

$$\|f\|_{MK^{\beta,\lambda}_{\ell,q,\omega}(\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\beta\ell}\|f\chi_k\|^\ell_{L^q_\omega(\mathbb{Q}_p^n)}\Big)^{1/\ell}. \tag{2.6}$$

Let us recall to define the weighted central BMO p-adic space.

Definition 2.5. *Let $1 \le q < \infty$ and ω be a weight function. The central bounded mean oscillation space $C\dot{M}O^q_\omega(\mathbb{Q}_p^n)$ is defined as the set of all functions $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^n)$ such that*

$$\|f\|_{C\dot{M}O^q_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \Big(\frac{1}{\omega(B_\gamma)} \int_{B_\gamma} |f(x) - f_{\omega,B_\gamma}|^q\omega(x)dx\Big)^{\frac{1}{q}} < \infty, \tag{2.7}$$

where

$$f_{\omega,B_\gamma} = \frac{1}{\omega(B_\gamma)} \int\limits_{B_\gamma} f(x)\omega(x)dx.$$

It is well known that the theory of A_q weight was first introduced by Benjamin Muckenhoupt on the Euclidean spaces to characterise the boundedness of Hardy-Littlewood maximal functions on the weighted Lebesgue spaces (see [21] for further detail). For A_q weights on the p-adic fields, more generally, on the local fields or homogeneous type spaces, see [8, 15] for more details.

Definition 2.6. *Let $1 < \ell < \infty$. We say that a weight $\omega \in A_\ell(\mathbb{Q}_p^n)$ if there exists a constant C such that for all balls B,*

$$\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B \omega(x)^{-1/(\ell-1)}dx\Big)^{\ell-1} \leq C.$$

We say that a weight $\omega \in A_1(\mathbb{Q}_p^n)$ if there is a constant C such that for all balls B,

$$\frac{1}{|B|}\int_B \omega(x)dx \leq C \operatorname*{essinf}_{x\in B} \omega(x).$$

We denote by $A_\infty(\mathbb{Q}_p^n) = \bigcup\limits_{1\leq \ell<\infty} A_\ell(\mathbb{Q}_p^n)$.

Let us recall the following standard result related to the Muckenhoupt weights.

Proposition 2.7. (i) *$A_\ell(\mathbb{Q}_p^n) \subsetneq A_q(\mathbb{Q}_p^n)$, for $1 \leq \ell < q < \infty$.*

(ii) *If $\omega \in A_\ell(\mathbb{Q}_p^n)$ for $1 < \ell < \infty$, then there is an $\varepsilon > 0$ such that $\ell - \varepsilon > 1$ and $\omega \in A_{\ell-\varepsilon}(\mathbb{Q}_p^n)$.*

It is said that ω satisfies the reverse Hölder condition of order $r > 1$ (in symbols $\omega \in RH_r(\mathbb{Q}_p^n)$) iff there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)^r dx\right)^{1/r} \leq \frac{C}{|B|}\int_B \omega(x)dx,$$

for all balls $B \subset \mathbb{Q}_p^n$. By virtue of Theorem 19 and Corollary 21 in [16], we have $\omega \in A_\infty(\mathbb{Q}_p^n)$ if and only if there exists some $r > 1$ such that $\omega \in RH_r(\mathbb{Q}_p^n)$. Moreover, if $\omega \in RH_r(\mathbb{Q}_p^n)$, $r > 1$, then $\omega \in RH_{r+\varepsilon}(\mathbb{Q}_p^n)$ for some $\varepsilon > 0$. We thus write $r_\omega = \sup\{r > 1 : \omega \in RH_r(\mathbb{Q}_p^n)\}$ to denote the critical index of ω for the reverse Hölder condition.

To end this section, let us give some standard properties of A_ℓ weights which they are proved in the similar way as the setting (see Proposition 2.4 and Proposition 2.5 in [22] for more details).

Proposition 2.8. *If $\omega \in A_\ell(\mathbb{Q}_p^n)$, $1 \leq \ell < \infty$, then for any $f \in L^1_{\rm loc}(\mathbb{Q}_p^n)$ and any ball $B \subset \mathbb{Q}_p^n$,*

$$\frac{1}{|B|}\int_B |f(x)|dx \leq C\left(\frac{1}{\omega(B)}\int_B |f(x)|^\ell \omega(x)dx\right)^{1/\ell}.$$

Proposition 2.9. *Let $\omega \in A_\ell(\mathbb{Q}_p^n) \cap RH_r(\mathbb{Q}_p^n)$, $\ell \geq 1$ and $r > 1$. Then, there exist constants $C_1, C_2 > 0$ such that*

$$C_1\left(\frac{|E|}{|B|}\right)^\ell \leq \frac{\omega(E)}{\omega(B)} \leq C_2\left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E of a ball B.

3. THE MAIN RESULTS

Let first us give the boundedness for the commutators of p-adic Hardy-Cesàro operators on the Morrey p-adic spaces with the power weight.

Theorem 3.1. *Let $1 \le q < \infty$, $1 < q_1, r_1 < \infty$ such that $1/q = 1/q_1 + 1/r_1$, and $1/q_1 < \lambda < 0$. Let $b \in C\dot{M}O^{r_1}_{\omega}(\mathbb{Q}_p^n)$ and $\omega(x) = |x|_p^{\gamma}$ for $\gamma > -n$. If*

$$\mathcal{A}_1 = \int_{\mathbb{Z}_p^*} |s(y)|_p^{(n+\gamma)\lambda} \psi(y) |\varphi(y)| dy < \infty,$$

where

$$\psi(y) = 1 + |\log_p |s(y)|_p|,$$

then $\mathcal{C}^{p,b}_{\varphi,s}$ is bounded from $\dot{B}^{q_1,\lambda}_{\omega}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)$.

Proof. For simplicity of notation, we denote $\psi_1(y) = |s(y)|_p^{-\frac{(n+\gamma)}{q_1}} \psi(y)$. Let first us prove the following inequality

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{L^q_\omega(B_\eta)} \lesssim \|b\|_{C\dot{M}O^{r_1}_{\omega}(\mathbb{Q}_p^n)} p^{\frac{\eta(n+\gamma)}{r_1}} \int_{\mathbb{Z}_p^*} |\varphi(y)| \psi_1(y) \|f\|_{L^{q_1}_\omega(B_{\eta+m})} dy, \tag{3.1}$$

for any $\eta \in \mathbb{Z}$, where $m = \log_p |s(y)|_p$. Indeed, using the Minkowski inequality and the Hölder inequality, we have

$$\big\|\mathcal{C}^{p,b}_{\varphi,s}(f)\big\|_{L^q_\omega(B_\eta)} \lesssim \int\limits_{\mathbb{Z}_p^*} |\varphi(y)| \|b(\cdot) - b(s(y)\cdot)\|_{L^{r_1}_\omega(B_\eta)} \|f(s(y)\cdot)\|_{L^{q_1}_\omega(B_\eta)} dy. \tag{3.2}$$

Now we need to show that

$$\begin{aligned} &\|b(\cdot) - b(s(y)\cdot)\|_{L^{r_1}_\omega(B_\eta)} \\ &\lesssim p^{\frac{\eta(n+\gamma)}{r_1}} \Big(1 + \log_p |s(y)|_p \chi_{\{|s(y)|_p \ge p\}} - \log_p |s(y)|_p \chi_{\{|s(y)|_p \le 1\}}\Big) \big\|b\big\|_{C\dot{M}O^{r_1}_{\omega}(\mathbb{Q}_p^n)}. \end{aligned} \tag{3.3}$$

For simplicity of notation, we put

$$K_1 = \|b(\cdot) - b_{\omega,B_\eta}\|_{L^{r_1}_\omega(B_\eta)},$$

$$K_2 = \|b(s(y)\cdot) - b_{\omega,B_{\eta+m}}\|_{L^{r_1}_\omega(B_\eta)},$$

and

$$K_3 = \|b_{\omega,B_\eta} - b_{\omega,B_{\eta+m}}\|_{L^{r_1}_\omega(B_\eta)}.$$

Thus it is easy to see that

$$\big\|b(\cdot) - b(s(y)\cdot)\big\|_{L^{r_1}_\omega(B_\eta)} \le K_1 + K_2 + K_3. \tag{3.4}$$

It follows from the definition of the space $C\dot{M}O^{r_1}_{\omega}(\mathbb{Q}_p^n)$ that

$$K_1 \le \omega(B_\eta)^{\frac{1}{r_1}} \big\|b\big\|_{C\dot{M}O^{r_1}_{\omega}(\mathbb{Q}_p^n)} \lesssim p^{\frac{\eta(n+\gamma)}{r_1}} \big\|b\big\|_{C\dot{M}O^{r_1}_{\omega}(\mathbb{Q}_p^n)}. \tag{3.5}$$

Let next us estimate K_2. Using the formula for change of variables, one has

$$K_2 = \Big(\int\limits_{B_\eta} |b(s(y)x) - b_{\omega,B_{\eta+m}}|^{r_1} \omega(x) dx\Big)^{\frac{1}{r_1}}$$

$$= \Big(\int\limits_{B_{\eta+m}} |b(z) - b_{\omega,B_{\eta+m}}|^{r_1} |s(y)^{-1}z|_p^{\gamma} |s(y)^{-n}|_p dz \Big)^{\frac{1}{r_1}}$$

$$\leq |s(y)|_p^{-\frac{(n+\gamma)}{r_1}} \omega(B_{\eta+m})^{\frac{1}{r_1}} \Big(\frac{1}{\omega(B_{\eta+m})} \int\limits_{B_{\eta+m}} |b(z) - b_{\omega,B_{\eta+m}}|^{r_1} \omega(z) dz \Big)^{\frac{1}{r_1}}$$

$$\lesssim p^{\frac{\eta(n+\gamma)}{r_1}} \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)}. \tag{3.6}$$

By the Hölder inequality, for all $k \in \mathbb{Z}$, we also have

$$\begin{aligned} \big|b_{\omega,B_k} - b_{\omega,B_{k+1}}\big| &\leq \frac{1}{\omega(B_k)} \int\limits_{B_k} \big|b(x) - b_{\omega,B_{k+1}}\big| \omega(x) dx \\ &\leq \frac{\omega(B_{k+1})^{\frac{1}{r_1'}}}{\omega(B_k)} \Big(\int\limits_{B_{k+1}} \big|b(x) - b_{\omega,B_{k+1}}\big|^{r_1} \omega(x) dx \Big)^{\frac{1}{r_1}} \\ &\leq \frac{\omega(B_{k+1})}{\omega(B_k)} \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)} \lesssim \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)}. \end{aligned}$$

For $m \geq 1$, we get

$$\begin{aligned} \big|b_{\omega,B_\eta} - b_{\omega,B_{\eta+m}}\big| &\leq \big|b_{\omega,B_\eta} - b_{\omega,B_{\eta+1}}\big| + \cdots + \big|b_{\omega,B_{\eta+m-1}} - b_{\omega,B_{\eta+m}}\big| \\ &\lesssim m \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)} = \log_p |s(y)|_p \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)}. \end{aligned}$$

Otherwise,

$$\big|b_{\omega,B_\eta} - b_{\omega,B_{\eta+m}}\big| \lesssim -m \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)} = -\log_p |s(y)|_p \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)}.$$

Consequently,

$$\begin{aligned} K_3 &\leq \omega(B_\eta)^{\frac{1}{r_1}} \big|b_{\omega,B_\eta} - b_{\omega,B_{\eta+m}}\big| \\ &\lesssim \omega(B_\eta)^{\frac{1}{r_1}} \Big(\log_p |s(y)|_p \chi_{\{|s(y)|_p \geq p\}} - \log_p |s(y)|_p \chi_{\{|s(y)|_p \leq 1\}} \Big) \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)}. \end{aligned} \tag{3.7}$$

This, together with inequalities (3.4), (3.5) and (3.6), yields that the inequality (3.3) holds. On the other hand, it is not hard to show that

$$\|f(s(y)\cdot)\|_{L_{\omega}^{q_1}(B_\eta)} \leq |s(y)|_p^{\frac{-(n+\gamma)}{q_1}} \|f\|_{L_{\omega}^{q_1}(B_{\eta+m})}.$$

Combining this inequality with (3.2) and (3.3) above, the inequality (3.1) is proved.

Let us now give the proof of the theorem as follows. For any $\eta \in \mathbb{Z}$, by (3.1), one has

$$\frac{1}{\omega(B_\gamma)^{\frac{1}{q}+\lambda}} \|\mathcal{C}_{\varphi,s}^{p,b}(f)\|_{L_{\omega}^{q}(B_\eta)} \lesssim \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)} \left(\int_{\mathbb{Z}_p^*} |\varphi(y)| \psi_1(y) \frac{p^{\frac{\eta(n+\gamma)}{r_1}} \omega(B_{\eta+m})^{\frac{1}{q_1}+\lambda}}{\omega(B_\eta)^{\frac{1}{q}+\lambda}} dy \right) \|f\|_{\dot{B}_{\omega}^{q_1,\lambda}(\mathbb{Q}_p^n)}.$$

By the condition $1/q = 1/q_1 + 1/r_1$, we estimate

$$\frac{p^{\frac{\eta(n+\gamma)}{r_1}} \omega(B_{\eta+m})^{\frac{1}{q_1}+\lambda}}{\omega(B_\eta)^{\frac{1}{q}+\lambda}} \simeq \frac{p^{\frac{\eta(n+\gamma)}{r_1}} p^{(\eta+m)(n+\gamma)(\frac{1}{q_1}+\lambda)}}{p^{\eta(n+\gamma)(\frac{1}{q}+\lambda)}} = |s(y)|_p^{(n+\gamma)(\frac{1}{q_1}+\lambda)},$$

which implies

$$\|\mathcal{C}_{\varphi,s}^{p,b}(f)\|_{\dot{B}_{\omega}^{q,\lambda}(\mathbb{Q}_p^n)} \lesssim \mathcal{A}_1 \|b\|_{C\dot{M}O_{\omega}^{r_1}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}_{\omega}^{q_1,\lambda}(\mathbb{Q}_p^n)}.$$

Thus the proof of the theorem is finished. □

By virtue of Theorem 3.1, we immediately have the following result.

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. If*

$$\mathcal{B}_1 = \int_{\mathbb{Z}_p^*} |\varphi(y)||y|_p^{(n+\gamma)\lambda} \log_p \frac{p}{|y|_p} dy < \infty,$$

then

$$\|\mathcal{H}_\varphi^{p,b}\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n) \to \dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)} \lesssim \mathcal{B}_1 \|b\|_{C\dot{M}O_\omega^{r_1}(\mathbb{Q}_p^n)}.$$

Next, the boundedness for the commutators of p-adic Hardy-Cesàro operators on the Morrey p-adic spaces with the Muckenhoupt weight is given as follows.

Theorem 3.3. *Let $1 \leq q, q_1^*, r_1^*, \zeta < \infty$, $-1/q_1^* < \lambda < 0$ and $1 < \delta < r_\omega$. Let $b \in C\dot{M}O_\omega^{r_1^*}(\mathbb{Q}_p^n)$, and $\omega \in A_\zeta$ with the finite critical index r_ω for the reverse Hölder condition such that*

$$\frac{1}{q} > \Big(\frac{1}{q_1^*} + \frac{1}{r_1^*}\Big)\zeta \frac{r_\omega}{r_\omega - 1}. \tag{3.8}$$

If

$$\mathcal{A}_2 = \int_{\mathbb{Z}_p^*} \Big(|s(y)|_p^{n\zeta\lambda} \chi_{\{|s(y)|_p \leq 1\}} + |s(y)|_p^{\frac{n(\delta-1)\lambda}{\delta}} \chi_{\{|s(y)|_p \geq p\}}\Big) \psi(y) |\varphi(y)| dy < \infty,$$

where $\psi(y)$ is given in Theorem 3.1, then $\mathcal{C}_{\varphi,s}^{p,b}$ is bounded from $\dot{B}_\omega^{q_1^*,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)$.

Proof. First, we prove the following inequality

$$\|\mathcal{C}_{\varphi,s}^{p,b}(f)\|_{L_\omega^q(B_\eta)} \lesssim \|b\|_{C\dot{M}O_\omega^{r_1^*}(\mathbb{Q}_p^n)} \left(\int_{\mathbb{Z}_p^*} |\varphi(y)|\psi(y) \frac{\omega(B_\eta)^{\frac{1}{q}}}{\omega(B_{\eta+m})^{\frac{1}{q_1^*}}} \|f\|_{L_\omega^{q_1^*}(B_{\eta+m})} dy\right), \tag{3.9}$$

for any $\eta \in \mathbb{Z}$, where recall again $m = \log_p |s(y)|_p$. Indeed, by the condition (3.8), there exist r_1, q_1 such that

$$\frac{1}{q_1} > \frac{\zeta}{q_1^*} \frac{r_\omega}{r_\omega - 1}, \quad \frac{1}{r_1} > \frac{\zeta}{r_1^*} \frac{r_\omega}{r_\omega - 1}, \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q}. \tag{3.10}$$

Arguing as in Theorem 3.1, we also get the relation (3.2). We still denote $K_1 = \|b(\cdot) - b_{\omega,B_\eta}\|_{L_\omega^{r_1}(B_\eta)}$, $K_2 = \|b(s(y)\cdot) - b_{\omega,B_{\eta+m}}\|_{L_\omega^{r_1}(B_\eta)}$ and $K_3 = \|b_{\omega,B_\eta} - b_{\omega,B_{\eta+m}}\|_{L_\omega^{r_1}(B_\eta)}$ as in the proof of Theorem 3.1 above for convenience. By $r_1 < r_1^*$, this leads to

$$K_1 \leq \omega(B_\eta)^{\frac{1}{r_1}} \|b\|_{C\dot{M}O_\omega^{r_i}(\mathbb{Q}_p^n)} \leq \omega(B_\eta)^{\frac{1}{r_1}} \|b\|_{C\dot{M}O_\omega^{r_1^*}(\mathbb{Q}_p^n)}. \tag{3.11}$$

By (3.10), there exists $\beta_1 \in (1, r_\omega)$ satisfying $r_1^*/\zeta = r_1 \beta_1'$. Thus, by applying the Hölder inequality, the reverse Hölder condition and Proposition 2.8, we infer

$$\begin{aligned}
K_2 &\leq \Big(\int_{B_\eta} |b(s(y)x) - b_{\omega,B_{\eta+m}}|^{\frac{r_1^*}{\zeta}} dx\Big)^{\frac{\zeta}{r_1^*}} \Big(\int_{B_\eta} \omega(x)^{\beta_1} dx\Big)^{\frac{1}{\beta_1 r_1}} \\
&\lesssim |B_\eta|^{\frac{-\zeta}{r_1^*}} \omega(B_\eta)^{\frac{1}{r_1}} \Big(\int_{B_\eta} |b(s(y)x) - b_{\omega,B_{\eta+m}}|^{\frac{r_1^*}{\zeta}} dx\Big)^{\frac{\zeta}{r_1^*}} \\
&\leq |B_\eta|^{\frac{-\zeta}{r_1^*}} \omega(B_\eta)^{\frac{1}{r_1}} |s(y)|_p^{\frac{-n\zeta}{r_1^*}} \Big(\int_{B_{\eta+m}} |b(z) - b_{\omega,B_{\eta+m}}|^{\frac{r_1^*}{\zeta}} dz\Big)^{\frac{\zeta}{r_1^*}} \\
&\lesssim |B_\eta|^{\frac{-\zeta}{r_1^*}} \omega(B_\eta)^{\frac{1}{r_1}} |s(y)|_p^{\frac{-n\zeta}{r_1^*}} \frac{|B_{\eta+m}|^{\frac{\zeta}{r_1^*}}}{\omega(B_{\eta+m})^{\frac{1}{r_1^*}}} \Big(\int_{B_{\eta+m}} |b(z) - b_{\omega,B_{\eta+m}}|^{r_1^*} \omega(z) dz\Big)^{\frac{1}{r_1^*}}
\end{aligned}$$

$$\lesssim \omega(B_\eta)^{\frac{1}{r_1}} \|b\|_{C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}_p^n)}. \tag{3.12}$$

Besides that, by (3.7) and $r_1 < r_1^*$, we get

$$K_3 \lesssim \omega(B_\eta)^{\frac{1}{r_1}} \Big(\log_p|s(y)|_p \chi_{\{|s(y)|_p \geq p\}} - \log_p|s(y)|_p \chi_{\{|s(y)|_p \leq 1\}}\Big) \|b\|_{C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}_p^n)}.$$

Hence, by (3.11) and (3.12), we conclude

$$\|b(\cdot) - b(s(y)\cdot)\|_{L^{r_1}_\omega(B_\eta)} \lesssim \omega(B_\eta)^{\frac{1}{r_1}} \psi(y) \|b\|_{C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}_p^n)}. \tag{3.13}$$

By (3.10) and estimating as (3.12) above, we can show that

$$\left(\int_{B_\eta} |f(s(y)x)|^{q_1} \omega(x) dx\right)^{\frac{1}{q_1}} \lesssim \omega(B_\eta)^{\frac{1}{q_1}} \omega(B_{\eta+m})^{\frac{-1}{q_1^*}} \|f\|_{L^{q_1^*}_\omega(B_{\eta+m})}.$$

Consequently, by (3.2) and (3.13), the inequality (3.9) holds.

Now we are in a position to give the proof of the theorem. It follows from (3.9) that

$$\frac{1}{\omega(B_\eta)^{\frac{1}{q}+\lambda}} \|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{L^q_\omega(B_\eta)} \lesssim \|b\|_{C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}_p^n)} \left(\int_{\mathbb{Z}_p^*} |\varphi(y)| \psi(y) \left(\frac{\omega(B_{\eta+m})}{\omega(B_\eta)}\right)^\lambda dy\right) \|f\|_{\dot{B}^{q_1^*,\lambda}_\omega(\mathbb{Q}_p^n)},$$

for any $\eta \in \mathbb{Z}$. Next, by the condition $\lambda < 0$ and Proposition 2.9, we have

$$\Big(\frac{\omega(B_{\eta+m})}{\omega(B_\eta)}\Big)^\lambda \lesssim \begin{cases} \Big(\dfrac{|B_{\eta+m}|}{|B_\eta|}\Big)^{\zeta\lambda} \lesssim |s(y)|_p^{n\zeta\lambda}, & \text{if} \quad m \leq 0, \\ \\ \Big(\dfrac{|B_{\eta+m}|}{|B_\eta|}\Big)^{\frac{(\delta-1)\lambda}{\delta}} \lesssim |s(y)|_p^{\frac{n(\delta-1)\lambda}{\delta}}, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\Big(\frac{\omega(B_{\eta+m})}{\omega(B_\eta)}\Big)^\lambda \lesssim |s(y)|_p^{n\zeta\lambda} \chi_{\{|s(y)|_p \leq 1\}} + |s(y)|_p^{\frac{n(\delta-1)\lambda}{\delta}} \chi_{\{|s(y)|_p \geq p\}}.$$

Consequently,

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \lesssim \mathcal{A}_2 \|b\|_{C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1^*,\lambda}_\omega(\mathbb{Q}_p^n)}.$$

Therefore, Theorem 3.3 is completely proved. □

According to Theorem 3.3, we also have the following result.

Corollary 3.4. *Let the assumptions of Theorem 3.3 hold. If*

$$\mathcal{B}_2 = \int_{\mathbb{Z}_p^*} |y|_p^{n\zeta\lambda} \log_p \frac{1}{|y|_p} |\varphi(y)| dy < \infty,$$

then

$$\|\mathcal{H}^{p,b}_\varphi\|_{\dot{B}^{q_1^*,\lambda}_\omega(\mathbb{Q}_p^n) \to \dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \lesssim \mathcal{B}_2 \|b\|_{C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}_p^n)}.$$

Theorem 3.5. *Let $1 \leq \ell, q < \infty$, $1 < q_1, r_1 < \infty$ such that $1/q = 1/q_1 + 1/r_1$, $\lambda \geq 0$ and $\beta_1 = \beta + \frac{n+\gamma}{r_1}$. Let $b \in C\dot{M}O^{r_1}_\omega(\mathbb{Q}_p^n)$ and $\omega(x) = |x|_p^\gamma$ for $\gamma > -n$. Then, if*

$$\mathcal{A}_3 = \int_{\mathbb{Z}_p^*} |s(y)|_p^{\lambda - \beta_1 - \frac{(n+\gamma)}{q_1}} \psi(y) |\varphi(y)| dy < \infty,$$

we have $\mathcal{C}^{p,b}_{\varphi,s}$ is bounded from $MK^{\beta_1,\lambda}_{\ell,q_1,\omega}(\mathbb{Q}_p^n)$ to $MK^{\beta,\lambda}_{\ell,q,\omega}(\mathbb{Q}_p^n)$.

Proof. Using a similar argument as the inequality (3.1) above, we have

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\chi_k\|_{L^q_\omega(\mathbb{Q}^n_p)} \lesssim \|b\|_{C\dot{M}O^{r_1}_\omega(\mathbb{Q}^n_p)} p^{\frac{k(n+\gamma)}{r_1}} \int_{\mathbb{Z}^*_p} |\varphi(y)|\psi_1(y)\|f\chi_{k+m}\|_{L^{q_1}_\omega(\mathbb{Q}^n_p)}dy,$$

for all $k \in \mathbb{Z}$. Hence, by the condition $\beta_1 = \beta + \frac{n+\gamma}{r_1}$ and the Minkowski inequality, one has

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{MK^{\beta,\lambda}_{\ell,q,\omega}(\mathbb{Q}^n_p)} = \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\beta\ell}\|\mathcal{C}^{p,b}_{\varphi,s}(f)\chi_k\|^\ell_{L^q_\omega(\mathbb{Q}^n_p)}\Big)^{1/\ell}$$
$$\le \|b\|_{C\dot{M}O^{r_1}_\omega(\mathbb{Q}^n_p)} \int_{\mathbb{Z}^*_p} |\varphi(y)|\psi_1(y) \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\beta_1\ell}\|f\chi_{k+m}\|^\ell_{L^{q_1}_\omega(\mathbb{Q}^n_p)}\Big)^{1/\ell}dy$$
$$\le \|b\|_{C\dot{M}O^{r_1}_\omega(\mathbb{Q}^n_p)} \int_{\mathbb{Z}^*_p} |\varphi(y)|\psi_1(y)p^{m(-\beta_1+\lambda)} \sup_{\eta_0\in\mathbb{Z}} p^{-\eta_0\lambda}\Big(\sum_{k=-\infty}^{\eta_0} p^{k\beta_1\ell}\|f\chi_{k}\|^\ell_{L^{q_1}_\omega(\mathbb{Q}^n_p)}\Big)^{1/\ell}dy.$$
$$\lesssim \mathcal{A}_3\|b\|_{C\dot{M}O^{r_1}_\omega(\mathbb{Q}^n_p)}\|f\|_{MK^{\beta_1,\lambda}_{\ell,q_1,\omega}(\mathbb{Q}^n_p)}.$$

Therefore, the proof of this theorem is finished. □

As a consequence, we immediately have the boundedness of $\mathcal{C}^{p,b}_{\varphi,s}$ on weighted Herz p-adic spaces.

Corollary 3.6. *Let the assumptions of Theorem 3.5 hold. Then, if*

$$\mathcal{A}_4 = \int_{\mathbb{Z}^*_p} |s(y)|_p^{-\beta_1-\frac{(n+\gamma)}{q_1}}\psi(y)|\varphi(y)|dy < \infty,$$

we have

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{K^{\beta,\ell}_{q,\omega}(\mathbb{Q}^n_p)} \lesssim \mathcal{A}_4\|b\|_{C\dot{M}O^{r_1}_\omega(\mathbb{Q}^n_p)}\|f\|_{K^{\beta_1,\ell}_{q_1,\omega}(\mathbb{Q}^n_p)}.$$

By Theorem 3.5, we also have the boundedness for commutators of the weighted Hardy-Littlewood operators on weighted Morrey-Herz p-adic spaces. Namely, the following is true.

Corollary 3.7. *Let the assumptions of Theorem 3.5 be fulfilled. If*

$$\mathcal{B}_3 = \int_{\mathbb{Z}^*_p} |y|_p^{\lambda-\beta_1-\frac{(n+\gamma)}{q_1}}\log_p\frac{p}{|y|_p}|\varphi(y)|dy < \infty,$$

then

$$\|\mathcal{H}^{p,b}_{\varphi}\|_{MK^{\beta_1,\lambda}_{\ell,q_1,\omega}(\mathbb{Q}^n_p)\to MK^{\beta,\lambda}_{\ell,q,\omega}(\mathbb{Q}^n_p)} \lesssim \mathcal{B}_3\|b\|_{C\dot{M}O^{r_1}_\omega(\mathbb{Q}^n_p)}.$$

Theorem 3.8. *Let $1 \le \ell, q, q_1^*, r_1^* < \infty$, $1 \le \zeta \le r_1^*$, $\beta \in \mathbb{R}$, $\beta_1^* < 0$, $b \in C\dot{M}O^{r_1^*}_\omega(\mathbb{Q}^n_p)$ and $\omega \in A_\zeta$ with the finite critical index r_ω for the reverse Hölder condition and $\delta \in (1, r_\omega)$. Assume that the hypothesis (3.8) in Theorem 3.1 is true and*

$$\frac{1}{q_1^*} + \frac{\beta_1^*}{n} = \frac{1}{q} + \frac{\beta}{n}. \tag{3.14}$$

(i) *If* $\frac{1}{q_1^*} + \frac{\beta_1^*}{n} \ge 0$ *and*

$$\mathcal{A}_5 = \int_{Z^*_p}\Big(|s(y)|_p^{-n\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\chi_{\{|s(y)|_p\le 1\}} + |s(y)|_p^{-n\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\chi_{\{|s(y)|_p\ge p\}}\Big)\psi(y)|\varphi(y)|dy < \infty,$$

then

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot{K}^{\beta,\ell,q}_{\omega}(\mathbb{Q}^n_p)} \lesssim \mathcal{A}_5\|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\|f\|_{\dot{K}^{\beta_1^*,\ell,q_1^*}_{\omega}(\mathbb{Q}^n_p)}.$$

(ii) *If* $\frac{1}{q_1^*}+\frac{\beta_1^*}{n}<0$ *and*

$$\mathcal{A}_6=\int_{Z_p^*}\Big(|s(y)|_p^{-n\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\chi_{\{|s(y)|_p\geq p\}}+|s(y)|_p^{-n\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\chi_{\{|s(y)|_p\leq 1\}}\Big)\psi(y)|\varphi(y)|dy<\infty,$$

then

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot{K}^{\beta,\ell,q}_{\omega}(\mathbb{Q}^n_p)} \lesssim \mathcal{A}_6\|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\|f\|_{\dot{K}^{\beta_1^*,\ell,q_1^*}_{\omega}(\mathbb{Q}^n_p)}.$$

Proof. Similarly to the proof for the inequality (3.9), for $k\in\mathbb{Z}$, we also have

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\chi_k\|_{L^q_\omega(\mathbb{Q}^n_p)} \lesssim \|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\int_{\mathbb{Z}_p^*}|\varphi(y)|\psi(y)\frac{\omega(B_k)^{\frac{1}{q}}}{\omega(B_{k+m})^{\frac{1}{q_1^*}}}\|f\|_{L^{q_1^*}_\omega(B_{k+m})}dy,$$

where $m=\log_p|s(y)|_p$. Thus, by using the Minkowski inequality and (3.14), we get

$$\begin{aligned}
&\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot{K}^{\beta,\ell,q}_{\omega}(\mathbb{Q}^n_p)}\\
&\lesssim \|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\Big(\sum_{k=-\infty}^{\infty}\omega(B_k)^{\beta\ell/n}\Big(\int_{Z_p^*}|\varphi(y)|\psi(y)\frac{\omega(B_k)^{\frac{1}{q}}}{\omega(B_{k+m})^{\frac{1}{q_1^*}}}\|f\|_{L^{q_1^*}_\omega(B_{k+m})}dy\Big)^{\ell}\Big)^{1/\ell}\\
&\leq \|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\Big(\int_{Z_p^*}|\varphi(y)|\psi(y)\Big(\sum_{k=-\infty}^{\infty}\Big(\frac{\omega(B_k)^{\frac{1}{q}+\frac{\beta}{n}}}{\omega(B_{k+m})^{\frac{1}{q_1^*}}}\|f\|_{L^{q_1^*}_\omega(B_{k+m})}\Big)^{\ell}\Big)^{1/\ell}dy\Big)\\
&= \|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\Big(\int_{Z_p^*}|\varphi(y)|\psi(y)\Big(\sum_{k=-\infty}^{\infty}\Big(\frac{\omega(B_k)^{\frac{1}{q_1^*}+\frac{\beta_1^*}{n}}}{\omega(B_{k+m})^{\frac{1}{q_1^*}}}\|f\|_{L^{q_1^*}_\omega(B_{k+m})}\Big)^{\ell}\Big)^{1/\ell}dy\Big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot{K}^{\beta,\ell,q}_{\omega}(\mathbb{Q}^n_p)} \lesssim \|b\|_{C\dot{M}O^{r_1^*}_{\omega}(\mathbb{Q}^n_p)}\Big(\int_{Z_p^*}|\varphi(y)|\psi(y)\times\\
&\times\Big(\sum_{k=-\infty}^{\infty}\Big\{\Big(\frac{\omega(B_k)}{\omega(B_{k+m})}\Big)^{\frac{1}{q_1^*}+\frac{\beta_1^*}{n}}\sum_{\eta=-\infty}^{m}\Big(\frac{\omega(B_{k+m})}{\omega(B_{k+\eta})}\Big)^{\frac{\beta_1^*}{n}}\omega(B_{k+\eta})^{\frac{\beta_1^*}{n}}\|f\chi_{k+\eta}\|_{L^{q_1^*}_\omega(\mathbb{Q}^n_p)}\Big\}^{\ell}\Big)^{1/\ell}dy\Big). \quad (3.15)
\end{aligned}$$

On the other hand, by Proposition 2.9 with $\beta_1^*<0$ and $\eta\leq m$, we have

$$\Big(\frac{\omega(B_{k+m})}{\omega(B_{k+\eta})}\Big)^{\frac{\beta_1^*}{n}} \lesssim \Big(\frac{|B_{k+m}|}{|B_{k+\eta}|}\Big)^{\frac{\beta_1^*(\delta-1)}{n\delta}}=p^{(m-\eta)\beta_1^*(\delta-1)/\delta}. \quad (3.16)$$

Now, by using Proposition 2.9, we consider the following two cases.

Case 1: $\frac{1}{q_1^*}+\frac{\beta_1^*}{n}\geq 0$. We get

$$\Big(\frac{\omega(B_k)}{\omega(B_{k+m})}\Big)^{\frac{1}{q_1^*}+\frac{\beta_1^*}{n}} \lesssim \begin{cases}\Big(\frac{|B_k|}{|B_{k+m}|}\Big)^{\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}=p^{-mn\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}=|s(y)|_p^{-n\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}, \text{ if } m\leq 0,\\[2ex] \Big(\frac{|B_k|}{|B_{k+m}|}\Big)^{\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}=p^{-mn\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\\ =|s(y)|_p^{-n\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}, \text{ otherwise.}\end{cases} \quad (3.17)$$

Case 2: $\frac{1}{q_1^*}+\frac{\beta_1^*}{n}<0$. We also get

$$\Big(\frac{\omega(B_k)}{\omega(B_{k+m})}\Big)^{\frac{1}{q_1^*}+\frac{\beta_1^*}{n}} \lesssim \begin{cases} \Big(\frac{|B_k|}{|B_{k+m}|}\Big)^{\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})} = p^{-mn\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})} \\ = |s(y)|_p^{-n\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}, \text{ if } m\le 0, \\ \\ \Big(\frac{|B_k|}{|B_{k+m}|}\Big)^{\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})} = p^{-mn\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})} = |s(y)|_p^{-n\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}, \text{ otherwise.} \end{cases} \tag{3.18}$$

To prove the part (i), by (3.15), (3.16) and (3.17), we have

$$\begin{aligned}\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot K^{\beta,\ell,q}_\omega(\mathbb{Q}_p^n)} \lesssim \|b\|_{C\dot MO^{r_1^*}_\omega(\mathbb{Q}_p^n)}\Big(\int_{Z_p^*}|\varphi(y)|\psi(y)\Big(|s(y)|_p^{-n\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\chi_{\{|s(y)|_p\le 1\}}\\ +|s(y)|_p^{-n\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\chi_{\{|s(y)|_p\ge p\}}\Big)\mathcal{T}(y)dy\Big),\end{aligned}$$

where

$$\mathcal{T}(y)=\Big(\sum_{k=-\infty}^{\infty}\Big\{\sum_{\eta=-\infty}^{m}p^{(m-\eta)\beta_1^*(\delta-1)/\delta}\omega(B_{k+\eta})^{\frac{\beta_1^*}{n}}\|f\chi_{k+\eta}\|_{L^{q_1^*}_\omega(\mathbb{Q}_p^n)}\Big\}^{\ell}\Big)^{1/\ell}.$$

By applying the Minkowski inequality again and $\beta_1^*<0$, we have

$$\begin{aligned}\mathcal{T}(y)&\le\sum_{\eta=-\infty}^{m}p^{(m-\eta)\beta_1^*(\delta-1)/\delta}\Big\{\sum_{k=-\infty}^{\infty}\Big(\omega(B_{k+\eta})^{\frac{\beta_1^*}{n}}\|f\chi_{k+\eta}\|_{L^{q_1^*}_\omega(\mathbb{Q}_p^n)}\Big)^{\ell}\Big\}^{1/\ell}\\ &\lesssim\|f\|_{\dot K^{\beta_1^*,\ell,q_1^*}_\omega(\mathbb{Q}_p^n)}.\end{aligned}$$

Thus we get

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot K^{\beta,\ell,q}_\omega(\mathbb{Q}_p^n)}\lesssim\mathcal{A}_5\|b\|_{C\dot MO^{r_1^*}_\omega(\mathbb{Q}_p^n)}\|f\|_{\dot K^{\beta_1^*,\ell,q_1^*}_\omega(\mathbb{Q}_p^n)}.$$

This shows that the part (i) is proved.

Similarly, by making (3.15), (3.16), (3.18) and estimating as above, we also have

$$\|\mathcal{C}^{p,b}_{\varphi,s}(f)\|_{\dot K^{\beta,\ell,q}_\omega(\mathbb{Q}_p^n)}\lesssim\mathcal{A}_6\|b\|_{C\dot MO^{r_1^*}_\omega(\mathbb{Q}_p^n)}\|f\|_{\dot K^{\beta_1^*,\ell,q_1^*}_\omega(\mathbb{Q}_p^n)}.$$

Theorem 3.8 is proved. □

Corollary 3.9. *Let $1\le\ell,q,q_1^*,r_1^*<\infty$, $1\le\zeta\le r_1^*$, $\beta\in\mathbb{R}$, $\beta_1^*<0$, $b\in C\dot MO^{r_1^*}_\omega(\mathbb{Q}_p^n)$ and $\omega\in A_\zeta$ with the finite critical index r_ω for the reverse Hölder condition and $\delta\in(1,r_\omega)$. Assume that the inequality (3.8) in Theorem 3.3 and the relation (3.14) in Theorem 3.8 are true.*

(i) *If $\dfrac{1}{q_1^*}+\dfrac{\beta_1^*}{n}\ge 0$ and*

$$\mathcal{B}_5=\int_{\mathbb{Z}_p^*}|y|_p^{-n\zeta(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})}\log_p\frac{1}{|y|_p}|\varphi(y)|dy<\infty,$$

then

$$\|\mathcal{H}^{p,b}_{\varphi}\|_{\dot K^{\beta_1^*,\ell,q_1^*}_\omega(\mathbb{Q}_p^n)\to\dot K^{\beta,\ell,q}_\omega(\mathbb{Q}_p^n)}\lesssim\mathcal{B}_5\|b\|_{C\dot MO^{r_1^*}_\omega(\mathbb{Q}_p^n)}.$$

(ii) *If* $\frac{1}{q_1^*} + \frac{\beta_1^*}{n} < 0$ *and*

$$\mathcal{B}_6 = \int_{\mathbb{Z}_p^*} |y|_p^{-n\frac{(\delta-1)}{\delta}(\frac{1}{q_1^*}+\frac{\beta_1^*}{n})} \log_p \frac{1}{|y|_p} |\varphi(y)| dy < \infty,$$

then

$$\|\mathcal{H}_\varphi^{p,b}\|_{\dot{K}_\omega^{\beta_1^*,\ell,q_1^*}(\mathbb{Q}_p^n) \to \dot{K}_\omega^{\beta,\ell,q}(\mathbb{Q}_p^n)} \lesssim \mathcal{B}_6 \|b\|_{C\dot{M}O_\omega^{r_1^*}(\mathbb{Q}_p^n)}.$$

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